On anti-Ramsey numbers for complete bipartite graphs and the *Turán* function

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Abstract

Given two graphs G and H with $H \subseteq G$ we consider the anti-Ramsey function $AR(G,H)$ which is the maximum number of colors in any edge-coloring of G so that every copy of H receives the same color on at least one pair of edges. The classical Turán function for a graph G and family of graphs $\mathcal{F}$, written $ex(G,\mathcal{F})$, is defined as the maximum number of edges of a subgraph of G not containing any member of $\mathcal{F}$. We show that there exists a constant $c>0$ so that $AR(K_n,K_{s,t})-ex(K_n,K_{s,t})<cn$ and c depends only on s and t, which implies $AR(K_n,K_{s,t})\leq cn^{2-\frac{1}{s}}$, for $s\leq t$ by a result of Kővari, Sós, and Turán.

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1 Introduction

For basic graph theoretic terminology and definitions see Diestel [2]. For specific definitions, we follow [1]. Given a graph H and an edge-coloring c of H, we say that c is *rainbow* if no two edges of H receive the same color. Given a copy A of $K_{s,t}$ for $t\geq s$, let $X(A)$ and $Y(A)$ denote the parts of A of order s and t, respectively and call them the *interior* and the *exterior*. For any $l>s$, consider the set of vertices $U=\{u_1,\dots,u_l\}$, and let $T\subseteq\binom{U}{s}$, the set of s-tuples of U. Let $T=\{x_1,\dots,x_k\}$ where x_i is an s-tuple for $1\leq i\leq k$. If $x_1\cap x_2\neq\emptyset,\dots,x_{k-1}\cap x_k\neq\emptyset$, and S is a graph containing T so that S can be partitioned into k edge-disjoint copies $A_1,\dots,A_k$ of $K_{s,t}$ with $X(A_i)=x_i$ for $1\leq i\leq k$, then we call S a $K_{s,t}$*-string* of length k. Furthermore, if $x_1\cap x_k\neq\emptyset$, then we call S a $K_{s,t}$*-ring* of length k. If S is a $K_{s,t}$-string and there exists a vertex $x\notin\{x_1,\dots,x_k\}$, adjacent to $s-1$ vertices of x_1 and a vertex of x_k which is not in x_1, then we call S a $K_{s,t}$*-string-tie*.

Let G be a graph and $c:E(G)\to\mathbb{Z}$ a coloring of $E(G)$. A *representing graph* of c is a spanning subgraph L of G containing exactly one edge of each color of c.

Given a multigraph G we define the *edge-multiplicity* $m(G)$ as the maximum number of edges between two vertices x and y.

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The Turán function for a graph G and family of graphs $\mathcal{F}$, written $ex(G, \mathcal{F})$, is defined as the maximum number of edges of a subgraph of G not containing any member of $\mathcal{F}$.

The anti-Ramsey function for graphs G and $H \subseteq G$, written $AR(G, H)$, is the maximum number of colors in any edge-coloring of G so that every copy of H receives the same color on at least one pair of edges

2 Brief History

The anti-Ramsey function and its relation to the Turán function were studied by Erdős, Simonovits, and Sós in [3], where they showed that $AR(K_n, H) - ex(K_n, \mathcal{H}) = o(n^2)$ with $\mathcal{H} = \{H - e : e \in E(H)\}$. Since then, many authors have worked on determining the asymptotic order of $AR(G, H)$ (see [4] for example). We follow the investigation of Axenovich and Jiang [1] who were able to determine that $AR(K_n, K_{2,n}) = (\frac{\sqrt{t-2}}{2})n^{\frac{3}{2}} + O(n^{\frac{4}{3}})$ and $AR(K_{n,n}, K_{2,n}) = \sqrt{t-2}n^{\frac{3}{2}} + O(n^{\frac{4}{3}})$. We show that if we exclude all rainbow complete bipartite graphs of fixed order, we can extend the previous technique and produce a general upper bound that follows the result from [5]:

Theorem 2.1. *$ex(K_n, K_{s,t}) \leq cn^{2-\frac{1}{s}}$ where $s \leq t$ and c depends on s and t.*

3 Excluding Rainbow Complete Bipartite Graphs

The following proposition was shown in [3]:

Proposition 3.1. $ex(G, \mathcal{H}) + 1 \leq AR(G, \mathcal{H}) \leq ex(G, \mathcal{H})$

Proof. For the upper bound, any representing subgraph of an H-free coloring of $E(G)$ is a subgraph of G containing no H subgraph. The number of colors used in a representing graph is equal to the number of edges of the representing graph $= AR(G, H)$. However, this number of colors is also the number of edges avoiding H.

For the lower bound, we consider a subgraph G' in G that has $ex(G, \mathcal{H})$ edges which does not contain any member of $\mathcal{H}$ as a subgraph. Color the edges of G' using distinct colors. Color the rest of G by some other color (all same color). The resulting coloring contains no rainbow copy of H and uses $ex(G, \mathcal{H}) + 1$ colors. $\square$

The arguments in the following two lemmas are similar to the case when $s = 2$ which can be found in [1].

Lemma 3.2. *If c is a coloring of $E(K_n)$ with no rainbow $K_{s,t}$, then c does not contain a rainbow $K_{s,t-1}$-string-tie.*

Proof. Let M be a rainbow $K_{s,t-1}$ string-tie in c that is of minimum length. Let the interior $X = \{x_1, \ldots, x_k\}$ where x_i are s-tuples and let the copies of $K_{s,t-1}$ that form P be labeled $B_1, \ldots, B_k$ where $X(B_i) = \{x_i\}$. Suppose M is obtained from a string P of length k by adding a vertex x (not in P) and making it adjacent to a vertex $u_k \in x_k \backslash x_1$, and the vertices $s_1 \subset x_1$ where $|s_1| = s - 1$. If $k = 2$, then M is a rainbow $K_{s,t}$. Let us assume that $k \geq 3$. Let $M_1 = B_1 \cup xs_1$ and

$M_2 = B_2 \cup \cdots \cup B_k \cup xu_k$. Since P is rainbow, for any $u_2 \in x_2$, $c(u_2)$ cannot be used in both M_1 and M_2. Thus, xu_2 completes a rainbow $K_{s,t}$-string-tie with either M_1 or M_2 which is shorter than M and a contradiction. $\square$

Lemma 3.3. *If H is a graph not containing a $K_{s,t}$-string-tie, then H is not a $K_{s,t}$-ring.*

Proof. Suppose that H is a $K_{s,t}$-ring of length k, $X(H) = \{x_1, \ldots, x_k\}$ where x_i are s-tuples and all the copies of $K_{s,t}$ forming H are $B_1, \ldots, B_k$ with $B_i = \{x_i\}$. Suppose first that the $Y(B_i)$ are pairwise disjoint so that $|Y(H)| = kt > k(s-1)+1 \geq |X(H)|$. By the pigeonhole principle, there exists $w \in Y(H) \backslash X(H)$. Without loss of generality, assume $w \in Y(B_1)$. Note that for $s_1 \subset x_1$ where $|s_1| = s-1$ and a vertex $u_k \in x_k \backslash x_1$, $\bigcup_{i=2}^{k} B_i \cup \{ws_1, wu_k\}$ is a $K_{s,t}$ string-tie.

Next, assume that there exists $l_1 < l_2$ such that $Y(B_{l_1}) \cap Y(B_{l_2}) \neq \emptyset$. Without loss of generality, suppose $l_1 = 1$ and l_2 is as small as possible. Let $v \in Y(B_1) \cap Y(B_{l_2})$. Let $l_3 = \max\{i \in [k] : v \in Y(B_i)\}$. Notice that $l_3 \geq l_2$. By the above observation, we have $l_2 - 1 \geq 2$ and $l_3 \leq k-1$. Since the vertices in $X(H)$ are all distinct, v is not a member of at least one of $\{x_2, \ldots, (x_{l_2} \backslash x_{l_3+1})\}$ or $\{x_{l_3+1}, \ldots, x_k, (x_1 \backslash x_2)\}$. Without loss of generality, suppose the statement holds for the first set. By our choice of l_2, we have $v \notin Y(\bigcup_{i=2}^{l_2-1} B_i)$ and hence $v \notin \bigcup_{i=2}^{l_2-1} B_i$. $\bigcup_{i=2}^{l_2-1} B_i \cup \{vx_2, v(x_{l_2} \backslash x_{l_3+1})\}$ contains a $K_{s,t}$-string-tie. $\square$

The next lemma is the key step to generalizing beyond the exclusion of $K_{2,t}$.

Lemma 3.4. *If $G' \subseteq G$, $|G| = n$, and G' does not contain a $K_{s,t}$-ring where $t \geq s$, then $\|G\| \leq ex(G, K_{s,t}) + st(n-1)$.*

Proof. We argue by contradiction and assume that H is a maximal collection of pairwise edge-disjoint $K_{s,t}$ in G' and that H contains k copies of $K_{s,t}$. Note that H contains kst edges and that removing the edges of H from G' leaves no copies of $K_{s,t}$. Combining this observation with our assumption produces

$$ex(G, K_{s,t}) + st(n-1) < e(G') \leq ex(G, K_{s,t}) + kst$$

from which we see that $k > n-1$.

Next we construct a graph F so that $V(F) = V(G')$ and for every member A of H where $X(A) = \{u_{i_1}, \ldots, u_{i_s}\}$, we create the path $u_{i_1} \ldots u_{i_s}$ in F. We note that two such paths may intersect on at most $s-1$ vertices and produce no more than $s-2$ multiple edges of multiplicity 2. Thus, F is a loopless multigraph with $k(s-1)$ edges and edge multiplicity $m(F) \leq k$. For every pair of vertices of F with at least two edges, we delete all but one edge between those vertices. Since the maximum number of multiple edges is $k(s-2)$, we are left with at least k edges after the deletion. Notice that the resulting graph is simple with n vertices and $k > n-1$ edges, so it must contain a cycle C. The edges of C are incident to vertices $X(A)$ where A are members of H, and therefore a subgraph containing vertices of C forms a $K_{s,t}$-ring in G' which is a contradiction. $\square$

Theorem 3.5. *For $s \leq t$ there exists a constant c, so that*

$$AR(K_n, K_{s,t}) - ex(K_n, K_{s,t-1}) \leq cn$$

Proof. Given a $K_{s,t}$-free coloring c of $E(K_n)$ with $AR(K_n, K_{s,t})$ colors and a representing graph H of c, we apply lemmas 3.2-3.4 in sequence, to obtain the result. $\square$

The above theorem together with Theorem 2.1 immediately gives the following

Corollary 3.6. *$AR(K_n, K_{s,t}) \leq cn^{2-\frac{1}{s}}$ where c depends on s and t.*

Note: We can extend our result to $K_{n,n}$ by repeating the above argument as in [1], to produce $AR(K_{n,n}, K_{s,t}) \leq cn^{2-\frac{1}{s}}$ where c depends on s and t.

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